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On the Spherical Correction of Object-glasses.

By A. E. Conrady.

The performance of object-glasses and mirrors is influenced very considerably by unavoidable residuals of spherical aberration. The usual treatment of these residuals, and of spherical aberration in general, by purely geometrical methods, *i.e.* as causes of a more or less serious confusion of the "rays" forming the image, is decidedly unsatisfactory from the physical point of view, for the undulatory theory defines a perfect lens or mirror as one which brings all light from a luminous point to the focus *in the same phase of vibration*, or which, in other words, transforms the spherical waves sent out by the object into truly spherical waves converging towards the focus. From this point of view any defects in a lens or mirror can become sensible only in so far as they sensibly throw certain portions of the total light *out of phase* with other portions; and there can be no doubt that the differences of phase with which the different portions of the total light meet at the focus are the only true and absolute measure of the seriousness of any outstanding aberrations.

A considerable amount of knowledge of spherical aberration in this physical sense may be derived from the known peculiarities in the course of the "rays" of geometrical optics when we bear in mind that the latter are properly defined as normals of the wave surface.

Lenses which combine small angular aperture with small dimensions can frequently be so well corrected that either trigonometrical computation or experimental test shows no sensible departure of any ray from the desired direction towards the focus; and as the sphere is the only surface which has all its normals intersecting in one point, it follows, as a matter of course, that in such cases the physical requirement as to phase-relation is strictly fulfilled.

But in the majority of cases, and always in lens-systems of either wide angular aperture (microscope-objectives) or of large dimensions (telescope-objectives) it is impossible completely to free all zones of the system from spherical aberration in the usual geometrical sense. Such lenses are said to be affected with "spherical zones," and in such cases the phase-relations at the focus remain to be investigated.

In the simplest case, *i.e.* that of object glasses of the type usually found in telescopes, computation shows that when spherical correction is established in the usual way by uniting the paraxial and the marginal rays, all intermediate zones are "undercorrected," or cut the axis nearer the lens. From this and from the above definition of rays we can conclude that the extra-axial parts of waves refracted by such systems are in advance of the ideal spherical waves converging towards the

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common focus of the axial and marginal rays; for it will be seen by reference to fig. 1 that only a generating line like that drawn can produce a surface with an arrangement of normals such as the geometrical theory proves to exist.

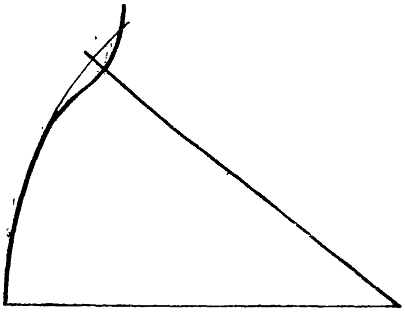

Fig. 1.

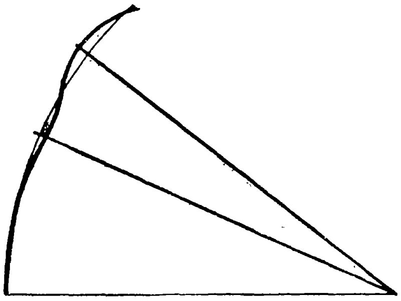

Fig. 2*a*.

In the case of lens-systems which contain deeply curved lenses, and in which the spherical residuals might attain prohibitive magnitude, the correction is sometimes carried a step farther by uniting the paraxial and the marginal rays with those passing through some selected intermediate zone. Further computation then again shows that all other zones are defective, and we can estimate that the real wave-form must be of the nature of those shown in fig. 2*a*, *b*, *c*, ; that it may indeed be any surface of rotation having the centre of curvature of its paraxial portion in the computed focus, and having two other zones the normals of which are directed towards the same focus. The most interesting result of the above simple and elementary considerations is probably this, that the zones for which spherical correction in the geometrical sense is established are apt to be the worst corrected in the physical sense. On the other hand the same considerations prove the desirability of practicable methods by which the *magnitude* of the existing differences of phase may be determined.

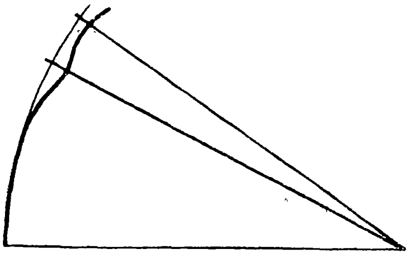

Fig. 2*b*.

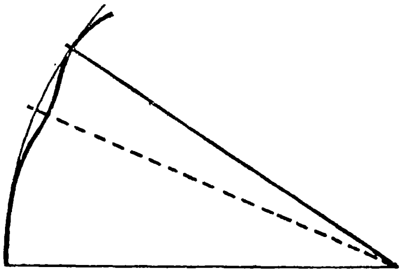

Fig. 2*c*.

I say *practicable* methods because it would be a simple trigonometrical task to compute the *total* path of a number of rays through any lens-system; indeed, it will be seen by reference

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to my paper on chromatic aberration* that the quantities D there used and trigonometrically defined represent successive portions of such paths; but as the total path is measured by the foot-rule and the wave-length by tenth-meters, it might occasionally require the hunting-up of a table of 10-figure logarithms to attain the necessary accuracy.

The formulæ to be sought for should therefore give the *difference* between different optical paths in terms which are reasonably small multiples of the wave-length.

The first method fulfilling this condition which I succeeded in devising more than ten years ago is briefly this:

Let $f \, O \, f'$ (fig. 3) be the optical axis of some lens-system; O P a refracting surface, f at the distance l from O a luminous point in the medium of refractive index n, to the left of the refracting surface; f' at the distance l' from O and in the medium of index n' the point for which we seek the difference of the optical paths of the axial ray $f \, O \, f'$ and the marginal ray $f \, P \, f'$, the point P being defined by coordinates x and y.

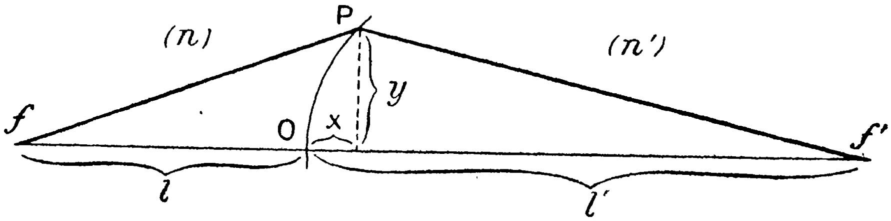

FIG. 3.

The coordinates shall be positive to the right of O and above the axis $f f'$, negative to the left of O and below $f f'$

Bearing this in mind, we deduce by inspection

$$\text{Path of axial ray} = l' - l$$

$$\text{Path of marginal ray} = \sqrt{(l'-x)^2 + y^2} - \sqrt{(l-x)^2 + y^2}$$

To obtain the *optical* paths we must multiply any distance by the index of the medium in which it is measured in order to reduce it to its *equivalent* distance in space; for the refractive index is inversely proportional to the velocity of light in any medium.

With this definition we can write down the difference between the optical paths of axial minus marginal rays as

$$\text{I.} \quad w = n'(l' - \sqrt{(l'-x)^2 + y^2}) - n(l - \sqrt{(l-x)^2 + y^2})$$

As we have made no restrictions either with regard to the refracting surface or the points f and f', this equation is quite general. If we put $w = 0$ it is the general equation of the

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* *Monthly Notices*, 1904 January, p. 186.

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generating line of a refracting surface which is aplanatic with regard to the conjugates *f* and *f′* (Cartesian ovals).

We will now restrict ourselves to a spherical surface of radius r; this gives us the well-known relation

$$y^2 = 2rx - x^2$$

between x and y. Introducing this for y^2 in I., and carrying out a few other obvious operations at the same time, we obtain

$$\text{I.*}\quad w = n'l'\left(1 - \sqrt{1 - 2\frac{x}{l'}\frac{l'-r}{l'}}\right) - nl\left(1 - \sqrt{1 - 2\frac{x}{l}\frac{l-r}{l}}\right)$$

putting

$$c' = \frac{x}{l'}\frac{l'-r}{l'} \qquad c = \frac{x}{l}\frac{l-r}{l}$$

and developing the square roots by McLaurin's theorem, we find

$$w = \begin{Bmatrix} n'l'(c' + \frac{1}{2}c'^2 + \frac{1}{2}c'^3 + \frac{5}{8}c'^4 + \frac{7}{8}c'^5 + \quad \dots \quad \dots \quad \dots) \\ -nl(c + \frac{1}{2}c^2 + \frac{1}{2}c^3 + \ \dots \quad \dots \quad \dots \quad \dots \quad \dots) \end{Bmatrix}$$

or separating the first-order terms

$$\text{II.}\quad w = n'l'c' - n.l.c + \begin{Bmatrix} n'l'c'^2(\frac{1}{2} + \frac{1}{2}c' + \frac{5}{8}c'^2 + \quad \dots \quad \dots) \\ -n.l.c^2(\frac{1}{2} + \frac{1}{2}c + \frac{5}{8}c^2 + \quad \dots \quad \dots) \end{Bmatrix}$$

The terms c' and c involve the abscissa x which in the case of a circle diminishes rapidly as P approaches the axis; hence for paraxial rays the first terms of II. will alone be sensible, giving ${}_0w = n'l'c' - nlc$.

If now we put ${}_0w = 0$ we obviously have the condition which makes *f′* the focus of the paraxial rays from *f*, for ${}_0w = 0$ means that all light passing from *f* through the paraxial zone arrives simultaneously at *f′* or becomes a spherical wave-train converging towards *f′*; and adopting this further restriction as to the points *f* and *f′* being paraxial conjugates, we have l' determined by the condition ${}_0w = 0$; for reintroducing the values of c and c' we obtain the condition

$$n'\frac{l'-r}{l'} = n\frac{l-r}{l} \text{ or } \frac{n'}{l'} = \frac{n}{l} + \frac{n'-n}{r}$$

which is arrived at in geometrical optics by a totally different method.

We now have for the focus of the paraxial rays thus determined

$$w = n'l'c'^2(\tfrac{1}{2} + \tfrac{1}{2}c' + \tfrac{5}{8}c'^2 + \ \dots\ ) - nlc(\tfrac{1}{2} + \tfrac{1}{2}c + \tfrac{5}{8}c^2 + \dots)$$

The rigorous numerical computation of this would complete our task. And this is *possible* because the c and c' are, in all practical cases, limited to a narrow range of values, from about

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$-1 \cdot 5$ to $+\cdot 3$; hence we can prepare a table * giving the numerical value of the round brackets for any practically possible c, really a very short and easily computed table. The series itself can be used for the small values of c usually occurring, whilst for the larger values it is easy to see, by referring back to the original equation, that we have

$$f_{(c)} = \tfrac{1}{2} + \tfrac{1}{2}c + \tfrac{5}{8}c^2 + \ldots = \frac{1 - \sqrt{1 - 2c'} - c}{c^2}$$

and that the value of the series may be obtained from this formula. Calling this tabulated value of the series $f_{(c)}$ we then have the simple formula for the difference of optical paths

$$\text{III.} \qquad w = n'l'c'^2 f_{(c')} - nlc^2 f_{(c)}$$

Up to this point the solution of the problem is a very simple and convenient one; but when applying it to a succession of refracting or reflecting surfaces, a troublesome correction has to be taken into account which arises from the presence of spherical aberration in the geometrical sense.

Equation III. is rigorously correct for rays directed towards the focus of the paraxial pencil, but as a rule the marginal rays do not travel in that direction, being deflected through spherical aberration; and thence arises the correction. Referring to fig. 4, where f' is assumed to be the focus of the paraxial pencil

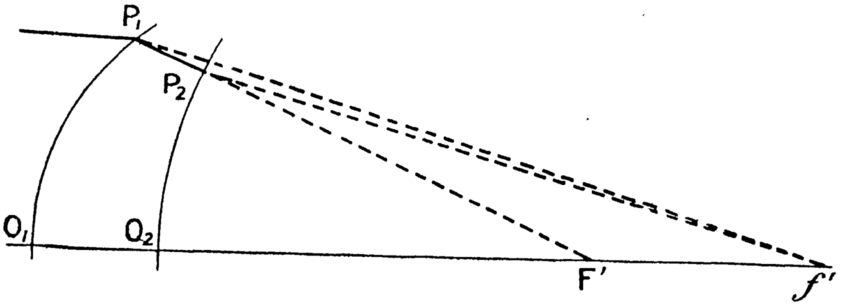

Fig. 4.

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* The following values of $\log f_{(c)}$ may be welcome to those who would like to compute by this process:

$c = -\cdot 10$	$\log f_{(c)} = 9\cdot 6585$	$c = \cdot 00$	$\log f_{(c)} = 9\cdot 6990$
$\cdot 09$	$9\cdot 6623$	$+\cdot 01$	$9\cdot 7033$
$\cdot 08$	$9\cdot 6661$	$\cdot 02$	$9\cdot 7078$
$\cdot 07$	$9\cdot 6701$	$\cdot 03$	$9\cdot 7123$
$\cdot 06$	$9\cdot 6740$	$\cdot 04$	$9\cdot 7169$
$\cdot 05$	$9\cdot 6780$	$\cdot 05$	$9\cdot 7215$
$\cdot 04$	$9\cdot 6821$	$\cdot 06$	$9\cdot 7263$
$\cdot 03$	$9\cdot 6862$	$\cdot 07$	$9\cdot 7311$
$\cdot 02$	$9\cdot 6904$	$\cdot 08$	$9\cdot 7360$
$\cdot 01$	$9\cdot 6947$	$\cdot 09$	$9\cdot 7410$
		$\cdot 10$	$9\cdot 7461$

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as refracted by the surface O_1P_1, F' the corresponding focus of the marginal rays refracted by the same surface, Equation III. assumes the light to proceed along the straight line P_1f'; according to the law of refraction the light takes the path P_1F', reaching the second surface at P_2.

Without entering into an extended discussion, it may be merely stated that the correction is found to be equal to the excess of the two sides $(P_1P_2+P_2f')$ of the long triangle P_1P_2f' over the third side P_1f', and that the computation of this correction was often found to be the most troublesome part of the task. Still, in the case of ordinary telescope-objectives of the usual proportion of aperture to focus, the correction is almost insensible, and may safely be neglected; and in this case Equation III. may therefore be regarded as a complete and convenient solution of the problem. It is different, however, in lens-systems such as are found in microscope-objectives. Here the correction occasionally amounts to many wave-lengths, and becomes an essential part of the whole difference of optical paths.

It only remains to be added that by substituting $(n+dn)$ for n and $(n'+dn')$ for n' in Equation I. and reducing in a manner analogous to that applied to I. an expression may be obtained which allows of the correction of chromatic aberration in a manner analogous to that treated of in my paper on chromatic correction. As the method given in that paper is more convenient, I will not give the modification in detail.

As the troublesome correction mentioned above arises because the marginal rays are not directed towards the focus of the paraxial rays which is the point of reference in Equation III., this correction would obviously disappear if the point of intersection of the marginal rays with the optical axis were taken instead.

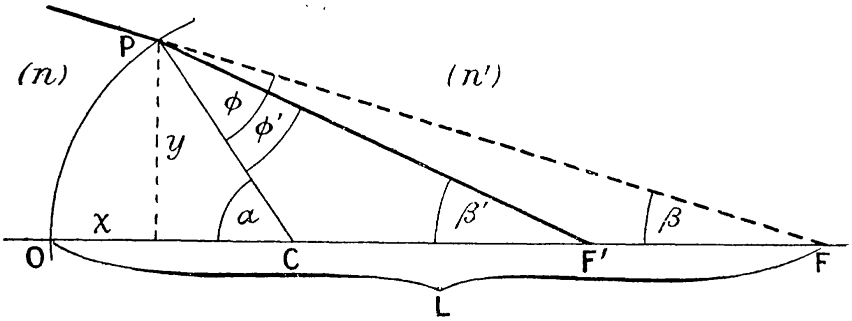

Fig. 5.

Realising this, I endeavoured for years to find a convenient solution of the problem on this basis, but until recently without success. As the path of a marginal ray can be conveniently defined and computed only by trigonometrical equations, it was obvious that the solution would have to be a trigonometrical one also.

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As is often the case, the finding of the elegant formula which follows depended on the choice of suitable variables and a "lucky" transformation.

The trigonometrical formulæ by which the course of rays through optical systems is defined were given in the former paper repeatedly quoted; referring to fig. 5 here reproduced, rays are defined by the abscissa OF = L and the angle of convergence β, and if the coordinates of the refracted ray are distinguished by a dash the relations between the quantities are

$$1. \qquad \sin\phi = \sin\beta\,\frac{L-r}{r}$$

$$2. \qquad \alpha = \beta+\phi$$

$$3. \qquad n'\sin\phi' = n\sin\phi$$

$$4. \qquad \beta' = \alpha-\phi'$$

$$5. \qquad L'-r = r\,\frac{\sin\phi'}{\sin\beta'}$$

We assume that the course of the rays to be considered has been computed by these equations, and that the quantities contained in them are therefore known.

Then we can obtain the difference of the optical paths of the marginal ray and of that proceeding along the optical axis which is produced by the refraction in the following manner.

Let a spherical wave, OA (fig. 6) in the medium of index n to the left of the refracting spherical surface OP be converging towards F; while the marginal portion is traversing the distance AP in the medium of index n, the axial part has entered the medium of index n' to the right of the refracting surface, but the *optical* paths of the two portions are necessarily the same. As the marginal ray is refracted towards F′ we should, on the supposition of freedom from spherical aberration, have a new spherical wave, BP, with F′ as centre, and the axial ray *should* have reached B in the second medium at the instant when the marginal ray reaches the refracting surface at P; the optical path OB on the axis should therefore be equal to that AP of

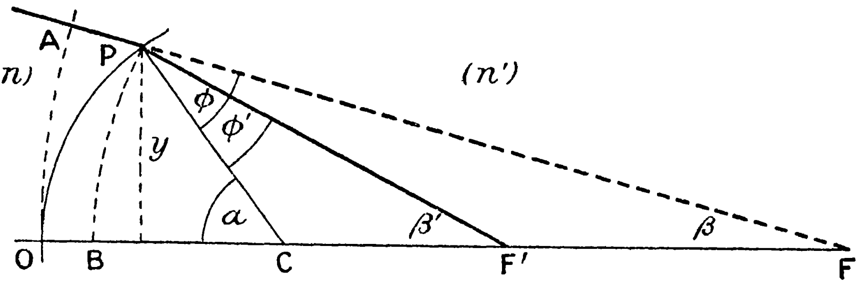

Fig. 6.

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the marginal ray; hence if these two optical paths are separately computed and found to differ, the difference is due to, and is a measure of, the spherical aberration produced by the surface OP, and is therefore the quantity we are trying to determine.

To obtain an expression for the length of AP we transfer it to the optical axis by a circle, PP′, with F as centre, so that

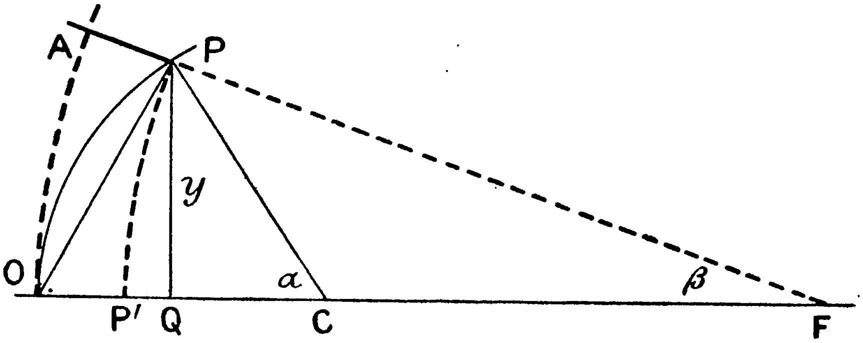

Fig. 7.

OP′ = AP (fig. 7). Dropping a perpendicular y from P upon the optical axis, we see that OP′ = OQ—P′Q.

Drawing OP, we have in the isosceles triangle OCP the angle at O $= 90° - \frac{\alpha}{2}$; hence in the right-angled triangle OQP

$$OQ = y \tan \frac{\alpha}{2}$$

Similarly we find

$$P'Q = y \tan \frac{\beta}{2}.$$

hence

$$OP' = AP = y \left(\tan \frac{\alpha}{2} - \tan \frac{\beta}{2} \right)$$

Applying the same process in the determination of the length of OB we find

$$OB = y \left(\tan \frac{\alpha}{2} - \tan \frac{\beta'}{2} \right)$$

Multiplying the two paths by the respective refractive indices, we obtain the optical paths; their difference W is therefore

$$\text{IV.} \quad W = n'y \left(\tan \frac{\alpha}{2} - \tan \frac{\beta'}{2} \right) - ny \left(\tan \frac{\alpha}{2} - \tan \frac{\beta}{2} \right)$$

This equation may be further reduced in a few steps by the following transformations:

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Replacing tan by $\frac{\sin}{\cos}$ in Equation IV. we obtain by cross-multiplication

$$W = n'y\frac{\sin\frac{\alpha-\beta'}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta'}{2}} - ny\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}$$

by (4) $\frac{\alpha-\beta'}{2} = \frac{\phi'}{2}$ by (2) $\frac{\alpha-\beta}{2} = \frac{\phi}{2}$

hence

$$W = n'y\frac{\sin\frac{\phi'}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta'}{2}} - ny\frac{\sin\frac{\phi}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}$$

Multiplying the first term by $\frac{2\cos\frac{\phi'}{2}}{2\cos\frac{\phi'}{2}}$, the second by $\frac{2\cos\frac{\phi}{2}}{2\cos\frac{\phi}{2}}$, and applying the general formula $2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \sin\alpha$ we obtain

$$W = \frac{n'y\sin\phi'}{2\cos\frac{\alpha}{2}\cos\frac{\beta'}{2}\cos\frac{\phi'}{2}} - \frac{ny\sin\phi}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\phi}{2}}$$

and as by (3) $n'\sin\phi' = n\sin\phi$ we have

$$W = \frac{n'y\sin\phi'}{2\cos\frac{\alpha}{2}}\left\{\frac{1}{\cos\frac{\beta'}{2}\cos\frac{\phi'}{2}} - \frac{1}{\cos\frac{\beta}{2}\cos\frac{\phi}{2}}\right\}$$

or

$$W = \frac{n'y\sin\phi'}{2\cos\frac{\alpha}{2}}\cdot\frac{\cos\frac{\beta}{2}\cos\frac{\phi}{2} - \cos\frac{\beta'}{2}\cos\frac{\phi'}{2}}{\cos\frac{\beta}{2}\cos\frac{\beta'}{2}\cos\frac{\phi}{2}\cos\frac{\phi'}{2}}$$

In the *numerator* of this we introduce $\cos\frac{\beta}{2} = \cos\left(\frac{\beta-\beta'}{2}+\frac{\beta'}{2}\right)$, an obvious identity, and resolve this by the usual formula. Similarly we introduce $\cos\frac{\phi'}{2} = \cos\left(\frac{\beta-\beta'}{2}+\frac{\phi}{2}\right)$, which follows by combining (2) and (4), which give $\beta+\phi = \beta'+\phi' = \alpha$, and resolve this also. We then obtain the numerator of the second fraction in the last equation for W in the form of four terms,

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two of which cross out, whilst the other two combine, with this result:

$$\text{V.}\qquad W=\frac{n'\,.\,y\,.\,\sin\phi'\sin\frac{\beta-\beta'}{2}\sin\frac{\phi-\beta'}{2}}{2\cos\frac{a}{2}\cos\frac{\beta}{2}\cos\frac{\phi}{2}\cos\frac{\beta'}{2}\cos\frac{\phi'}{2}}$$

This remarkable equation is undoubtedly the simplest and clearest *rigorous* expression to which the problem of spherical aberration has ever been reduced, and its discussion is therefore as interesting as it is easy.

As the angles occurring in V. are usually small, we see that the important terms are those of the numerator, the denominator being more in the nature of a correction.

If we now discuss the effect of variation of any of the five terms in the numerator, the first, n', tells us that, other things being equal, the aberration is proportional to the refractive index. The second term, similarly, makes the aberration proportional to the semi-aperture y; hence we can at once generalise that in similarly constructed optical systems the residual aberrations are proportional to the scale to which the different systems are constructed, and therefore that residuals which would be immaterial in a small object-glass may easily become serious in a similar object-glass of large size.

The term $\sin\phi'$ next shows that the aberration is approximately proportional to the angle of refraction, and the following term that it is also nearly proportional to the deflection produced by the refraction; these are both results which seem almost foregone conclusions, and call for no special notice. But now we come to the last term, and in this are wrapped up some of the chief mysteries of object-glass construction; for this term can become zero when all the other quantities have finite **values**, and when the course of the rays is therefore altered by refraction; it thus expresses the existence of points for which the sphere is free from spherical aberration.

Let us discover these points.

$\frac{\phi-\beta'}{2}$ becomes zero when $\phi=\beta'$, which implies $\sin\phi=\sin\beta'$, and, further, by (3) $\frac{n'}{n}\sin\phi'=\sin\beta'$. The vanishing of the last term of V. therefore depends on the condition $\frac{\sin\beta'}{\sin\phi'}=\frac{n'}{n}$.

If we now refer again to fig. 5, the trigonometrical sine-law applied to the triangle PCF′ gives us

$$\frac{\sin\beta'}{\sin\phi'}=\frac{r}{L'-r}=\frac{n'}{n},\ \text{or}\ L'-r=r\frac{n}{n'},\ \text{or}\ (6)L'=r\frac{n'+n}{n'}$$

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Further, by (2) and (4) we have $\phi - \beta' = \phi' - \beta$, or, proceeding as before,

$$\sin \phi' = \sin \beta = \frac{n}{n'} \sin \phi, \text{ or } \frac{\sin \beta}{\sin \phi} = \frac{n}{n'}$$

whence, by the triangle PCF,

$$\frac{\sin \beta}{\sin \phi} = \frac{n}{n'} = \frac{r}{\mathrm{L}-r}; \ \mathrm{L}-r = r\frac{n'}{n}; \ (7)\, \mathrm{L} = r\,\frac{n'+n}{n}$$

whilst $\sin \phi = \sin \beta'$, combined with $\frac{n}{n'} \sin \phi = \sin \beta$, both equations occurring in the above, give

$$(8)\ \frac{\sin \beta}{\sin \beta'} = \frac{n}{n'} = \text{a constant,}$$

to which we may add, by combining (6), (7), and (8):

$$(8^*)\ \frac{\mathrm{L}'}{\mathrm{L}} = \frac{n}{n'} = \frac{\sin \beta}{\sin \beta'}$$

By (6) and (7) the position of the pair of aplanatic points is defined. Equation 8 gives the important further information that these are aplanatic points in the wider sense of Professor Abbe, for they fulfil the optical sine-condition. The construction of wide-angled microscope-objectives depends on this property of the sphere, for only by taking advantage of these aplanatic points is it possible to deal with cones of light of as much as 140° angular aperture without introducing overwhelming amounts of spherical aberration.

On the other hand we now see that when ϕ and β have opposite signs the last term in V. tends to make the spherical aberration particularly heavy; this also is taken advantage of in object-glass construction when it is desired to obtain a large amount of spherical aberration—generally in the sense of *over*-correction—with only a moderate alteration of convergence. This discussion might be carried much further, but it may here suffice to add that if the rays converge towards points nearer the centre of curvature of the refracting surface than those defined by (6) and (7) above, we obtain refraction in the sense of that produced by a convex lens accompanied by spherical aberration in the sense of that of an ordinary concave lens (over-correction), a little known fact.

Equation V. has been deduced on the assumption of a spherical impinging wave; it will, however, be clear that any retardation or acceleration of the arriving marginal ray would be simply added to that produced by the refraction, and, further, that the total difference of optical paths of a marginal and the axial ray in any optical system will be the algebraical sum of all differences arising at successive surfaces.

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Referring once more to fig. 1 it will be seen that these new equations make it possible to compute the distance between the real refracted wave and the ideal one when spherical correction in the geometrical sense has been established and thus to obtain a trustworthy indication of the residual aberration; but an even more important application of the new equations when so used in conjunction with the usual trigonometrical computation lies in the higher correction of spherical aberration by adding to the usual geometrical condition, that the lens-system shall be so corrected that the marginal rays shall be directed towards the focus of the paraxial pencil, the further one *that the marginal rays shall also meet the paraxial ones without any difference of optical paths,* i.e. *in the same phase of vibration.*

This condition, which I believe to be quite original with me, is one which I have adopted as a definite standard in carrying spherical correction to higher perfection in all cases which call for it.

Equation IV. gives the difference between the marginal and axial paths; it may therefore be differentiated to find its variation with the wave-length—in other words the chromatic aberration—after the manner introduced by me in the paper quoted above. We obtain

$$\text{IV.*}\quad \frac{\partial W}{\partial \lambda}=\frac{\partial n'}{\partial \lambda}\cdot y\left(\tan\frac{\alpha}{2}-\tan\frac{\beta'}{2}\right)-\frac{\partial n}{\partial \lambda}y\left(\tan\frac{\alpha}{2}-\tan\frac{\beta}{2}\right)$$

Really α, β, and β' are also variable with λ, but I have proved in the former paper (and it may be accepted immediately as a deduction from Fermat's theorem of the minimum optical path) that the contributions of these to $\frac{\partial W}{\partial \lambda}$ add up to zero, and that they need not therefore be taken into account. Hence IV.* is an alternative equation for the computation of lens-systems in such a way as to attain minimum focus for a prescribed wave-length. It may be modified in a manner similar to that applied to IV., giving

$$\text{VI.}\quad \frac{\partial W}{\partial \lambda}=\frac{\partial n'}{\partial \lambda}y\frac{\sin\frac{\phi'}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta'}{2}}-\frac{\partial n}{\partial \lambda}y\frac{\sin\frac{\phi}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}$$

an expression which is at least as convenient as that given in the former paper, and which may advantageously be computed as a check.

By computing by the trigonometrical formulæ (1) to (5), and also by V. and VI., we may free a lens-system from spherical and chromatic aberration, although only one single ray has been followed through the system. This seems rather a startling proposition, and might at first sight seem simply absurd; it becomes

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acceptable when we consider that the method of optical paths has *one* optical path given without computation, *i.e.* the optical axis itself. This is perhaps one of the most striking illustrations of the power of these new methods of computation.

Further interesting results follow on reducing V. to first-order terms, *i.e.* applying it to rays sufficiently near the optical axis to make the sines equal to their angles, and the cosines equal to unity. Adopting the nomenclature of the former paper, *i.e.* a left-hand index o for these paraxial angles, we obtain easily

$$ {}_{\circ}W = \tfrac{1}{8} n' {}_{\circ}y {}_{\circ}\phi ({}_{\circ}\beta - {}_{\circ}\beta')({}_{\circ}\phi - {}_{\circ}\beta') $$

It is not difficult to see by referring to fig. 5 that ${}_{\circ}\beta = \frac{{}_{\circ}y}{l}$; ${}_{\circ}a = \frac{{}_{\circ}y}{r}$; ${}_{\circ}\beta' = \frac{{}_{\circ}y}{l'}$ from which by (2) and (4) ${}_{\circ}\phi$ and ${}_{\circ}\phi'$ may be found ; introducing these we find

$$ \text{VII.} \quad {}_{\circ}W = \tfrac{1}{8} n' \frac{{}_{\circ}y^4}{r^2} \cdot \frac{l' - r}{l'} \cdot \frac{l' - l}{ll'} \cdot \frac{ll' - rl' - rl}{ll'} $$

which shows that in the first approximation the differences of phase increase with the fourth power of the aperture. Being algebraical, Equation VII. may possibly supply an alternative analytical solution of the problem of computing a spherically corrected object-glass. The corresponding chromatic equation follows from VI., viz. :

$$ \frac{\partial {}_{\circ}W}{\partial \lambda} = \frac{\partial n'}{\partial \lambda} \quad \frac{{}_{\circ}y}{2} {}_{\circ}\phi' - \frac{\partial n}{\partial \lambda} \quad \frac{{}_{\circ}y}{2} {}_{\circ}\phi $$

or, eliminating the angles as above,

$$ \text{VIII.} \quad \frac{\partial {}_{\circ}W}{\partial \lambda} = \frac{\partial n'}{\partial \lambda} \quad \frac{{}_{\circ}y^2}{2} \quad \frac{l' - r}{l'r} - \frac{\partial n}{\partial \lambda} \quad \frac{{}_{\circ}y^2}{2} \quad \frac{l - r}{lr} $$

The differences of phase at focus due to chromatic aberration increase therefore as the square of the aperture.

In conclusion I will use VII. to show the connexion between spherical aberration regarded as a difference of phase and geometrical spherical aberration regarded as a departure of "rays" from the desired direction.

Reference to fig. 1 will make it quite clear that the two stand in the relation of a function and its differential coefficient, *i.e.* that if we have a function expressing the connexion between the ordinates and the distance between ideal and real waves, the differential coefficient of that function expresses the angle between the respective normals, which is a measure of the geometrical aberration.

Before making use of this interesting relation, we must remember that our equations throughout represent *optical* paths ;

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in order to get corresponding geometrical distances we must therefore divide them by the index of the medium, which in the case of VII. amounts to leaving out the factor n'. Bearing this in mind and putting the angle between the real and the ideal "ray" as ψ we deduce

$$\psi = \frac{\partial_0 W}{\partial_0 y} = \tfrac{1}{2}\frac{{}_0y^3}{r^2}\,\frac{l'-r}{l'}\cdot\frac{l'-l}{ll'}\cdot\frac{ll'-rl'-rl}{ll'}$$

and combining this with VII., we obtain the interesting relation

$$\frac{{}_0W}{\psi} = \frac{n'\,{}_0y}{4} \text{ or IX. } \psi = \frac{4\,{}_0W}{n'\,{}_0y}$$

By this equation we can approximately compute ψ from ${}_0W$, or *vice versa* ; as it stands it of course gives ψ in radians and ${}_0W$ in the same unit of length in which y is measured. To obtain ψ in seconds of arc we must multiply the right-hand side with 206,265. If y, the semiaperture, is measured in inches, we may express ${}_0W$ in wave-lengths by dividing the right-hand side by 50,000, the approximate number of average light-waves in one inch ; hence our equation for y in inches, ψ in seconds, and ${}_0W$ in approximate wave-lengths becomes, near enough,

$$\psi = 16\,\frac{{}_0W}{n'\,{}_0y}$$

which means that in an object-glass of two inches aperture ($y = 1$) one wave-length of physical aberration corresponds to 16 seconds of geometrical aberration, or a "circle of confusion" of 32 seconds.

Lord Rayleigh has shown that ${}_0W$ may become $\frac{1}{4}$ wave-length without serious deterioration of the image. Accepting this, we deduce for a 1-inch object-glass

$$\psi = \frac{16 \times \frac{1}{4}}{\frac{1}{2}} = 8 \text{ seconds}$$

or a circle of confusion of 16 seconds of arc as the geometrical aberration permissible. As the resolving power of an inch object-glass is about $4\frac{1}{2}$ seconds, this shows most clearly how utterly misleading the geometrical theory is. According to the latter we should have stars spread out into discs nearly four times as large as the resolving power ; in reality the image is little worse than that of a theoretically perfect glass.

This is really borne out by practical experience ; moderate amounts of spherical aberration are almost past detection at the focus ; they only show up when the star image is expanded by "racking in or out." It will nevertheless prove a great surprise to the vast majority of practical opticians who, impressed by the

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almost inconceivable smallness of a wave-length, consider it simply absurd to be asked to make the different paths equal within a fraction of this minute quantity; they would, however, be indignant if it were suggested that they could not bring the more familiar geometrical "rays" together within a circle of confusion corresponding in size to the theoretical resolving power of the object-glass. As I have just shown, it is far easier to attain Lord Rayleigh's quarter-wave standard than to reach the circle-of-confusion limit usually adopted in geometrical optics.

Bedford Park, W.:
1905 April 12.

On the General Design of Spectrographs to be attached to Equatorials of large aperture, considered chiefly from the point of view of Tremor-discs. By H. F. Newall.

The following note is an attempt to gain a general idea of the dimensions of that spectrographic installation that forms the turning-point at which increase in size is attended by increasingly disproportionate results. We know how, in the case of object-glasses, an increase in aperture is attended by an increase in absorption; and Vogel's numerical estimates (not to speak of others) have taught us that after the value of approximately nine feet for the diameter of an object-glass is passed, an increase in diameter would be attended by a diminution in brightness. In the case of spectrography we have for some time past recognised the need to use wide slits and high resolving powers. It is of interest to get even a rough estimate of the dimensions, for which, if we in any way design a spectrograph so that its resolving power is proportional to the aperture of the equatorial to which it is attached, the absorption in the spectrograph counterbalances the light-gathering power of the equatorial. The results will help us to decide along what lines of design lies the greatest promise of extension of power.

The paper divides itself roughly into three parts: (i.) a part dealing with the conveniences of estimating slit-widths in terms of visible diffractional phenomena instead of millimetres; (ii.) a part treating of the effects of atmospheric disturbance on star images—in short, of tremor-discs; (iii.) a section in which the attempt is made to reconcile the requirements of the astronomer (in the way of perfection of photographed spectra) with the disadvantageous conditions imposed by atmospheric disturbances.

We collect starlight under conditions that give us, when effects are summed up for a time, circular images or tremor-discs of not inconsiderable dimensions; we measure our photographs